FINITELY GENERATED MODULES OVER NOETHERIAN RINGS: INTERACTIONS BETWEEN ALGEBRA, GEOMETRY, AND TOPOLOGY

ZACHARY FLORES

My research covers three different aspects of commutative algebra, all with strong connections to geometry or topology. In a broad context, I study the category of finitely generated modules over graded and local Noetherian rings.

My current research goals are the following: The first is to compute higher K-groups of local Cohen-Macaulay rings, utilizing topological and categorical tools; the second is to understand Lefschetz properties for graded modules of finite length, through applications of algebraic geometry; and the third is to understand the algebraic structure of Macaulay duals of generic hyperplane arrangements. While these areas of commutative algebra may seem disparate, I want to stress that an underlying theme of my work is the development of new techniques that produce previously unknown examples from the tools that have revolutionized algebra and geometry. For example, in [15] I give a structure for $G_1(R)$, the first Quillen K-group of the category of finitely generated R-modules, for a large class of local Cohen-Macaulay rings, as well as providing numerous explicit computations of $G_1(R)$.

My research also illuminates a connection between Lefschetz properties in commutative algebra and vector bundles in algebraic geometry. Namely, in [14], we study syzygy bundles and their generic splitting types in an effort to give a large class of non-cyclic finite length modules which have the Weak Lefschetz Property, as well as generalizing classical results. Continuing this trend, I discuss how to lay the groundwork in [16] to study Lefschetz properties for a class of finite length modules that, when cyclic, are Artinian Gorenstein algebras, as well as how to study their syzygy bundles. Lastly, in [12], we explore the algebraic structure of the Macaulay dual of a generic hyperplane arrangement in $\mathbb{P}^r$.

1. Algebraic K-Theory for Cohen-Macaulay Rings

Throughout this section $(R, \mathfrak{m}, k)$ is a local Noetherian ring with unique maximal ideal $\mathfrak{m}$, and algebraically closed residue field $k := R/\mathfrak{m}$ of characteristic not two. Let $\mathcal{C}$ denote the category of finitely generated R-modules.

In [34], the *ith Quillen K-group of $\mathcal{C}$*, denoted by $K_i^Q(\mathcal{C})$, is defined to be the abelian group $\pi_{i+1}(BQ\mathcal{C}, 0)$, where $Q\mathcal{C}$ is Quillen's *Q-construction*, $BQ\mathcal{C}$ is the classifying space of $Q\mathcal{C}$, 0 is a fixed zero object, and π_{i+1} denotes the taking of a homotopy group. Write $G_i(R) := K_i^Q(\mathcal{C})$ and call $G_i(R)$ the *ith G-group of R*. While this definition is quite useful, it is very difficult to parse, and the groups $G_i(R)$ are hard to compute. In fact, $G_i(\mathbb{Z})$ is not known for all i and a significant work was spent in [25] to show $G_3(\mathbb{Z}) \cong \mathbb{Z}/48\mathbb{Z}$. In the spirit of calculating G-groups, the following is shown in [15]:

Theorem 1 (Theorem 3, [15])**.** Let $(R, \mathfrak{m}, k)$ be a local Cohen-Macaulay ring. Assume R is a Henselian k-algebra that admits a dualizing module and is also an isolated singularity. If the category of maximal Cohen-Macaulay R-modules, $\mathbf{mcm}\, R$, admits an n-cluster tilting object L such that $\operatorname{End}_R(L)^{\mathrm{op}}$ has finite global dimension, then there is a subgroup Ξ of $\operatorname{Aut}_R(L)_{\mathrm{ab}}$, the abelianization of $\operatorname{Aut}_R(L)_{\mathrm{ab}}$, and a free abelian group $\mathcal{H}$ such that

$$G_1(R) \cong \mathcal{H} \oplus \operatorname{Aut}_R(L)_{\mathrm{ab}}/\Xi$$

Theorem 1 generalizes (Theorem 2.12, [20]) and expands on work from [31, 32]. Theorem 1 contains several technical terms, but it should be stressed this broadens its applications, and proof of concept of this is provided by computing $G_1(R)$ for several families of singularities in [15]. An explicit example of when all conditions of Theorem 1 are satisfied is when R is a complete local Noetherian ring of finite Cohen-Macaulay type (Theorem 6, [26]), and all complete local Noetherian rings of finite Cohen-Macaulay type in dimension one are classified (Chapter 4, [27]).

While the definition of an n-cluster tilting object is central to the statement of Theorem 1, it is very technical. Important examples of when $\mathbf{mcm}\, R$ admits an n-cluster-tilting come from certain reduced hypersurface singularities in dimension one and three, where results from [9] and [22] show that an n-cluster tilting object not only exists, but its endomorphism ring has finite global dimension. In [15], $G_1(R)$ is computed for these classes of hypersurface singularities.

In [15], for all positive-dimensional cases in which $G_1(R)$ is fully computed, it is the case that $G_1(R)$ contains $\overline{R}^*$ (where $\overline{R}$ is the integral closure of R in its total quotient ring) as a direct summand. This leads to the following:

Question 1. If R satisfies all conditions of Theorem 1, does $G_1(R)$ contain $\overline{R}^*$ as a direct summand? Even more explicitly, with notation as in Theorem 1, is $\mathrm{Aut}_R(L)_{\mathrm{ab}}/\Xi \cong \overline{R}^*$?

Answering Question 1 would compute $G_1(R)$ explicitly for a large class of local Cohen-Macaulay rings, as well as providing techniques for the possible computation of higher G-groups.

Goal 1. Our current main interest is to show that if R satisfies all the hypotheses of Theorem 1, then $G_1(R) \cong \mathcal{H} \oplus \overline{R}^*$. If this proves to be too difficult, we can concentrate on the case when R is complete and of finite Cohen-Macaulay type, noting [15] provides several examples of computation in this direction.

To accomplish this goal, I aim to use ideas from [13] that involve applications of derived homological algebra and homotopy theory to algebraic K-theory. In particular, if $\mathcal{A}$ is the category of finitely generated left modules over $\mathrm{End}_R(L)^{\mathrm{op}}$, and it can be shown there is a Serre subcategory $\mathcal{B} \subseteq \mathcal{A}$, such that the quotient category $\mathcal{A}/\mathcal{B}$ is equivalent to the category of finitely generated modules over $\overline{R}$, Quillen's Localization Theorem (Theorem 5, [34]) may be applied to obtain a long exact sequence of G-groups which can further be analyzed with the intent of answering Question 1.

Throughout, we will want to see connections our work has with algebraic geometry, and we are curious about the recent work in this direction:

Question 2. We ask broadly if we can also use tools from geometry to aid in the computation of $G_i(R)$ for $i \geq 2$? For instance, there are approaches to the computations of higher K-groups that are largely geometric (see [29], for example), and we would like to understand how to apply these results in our setting.

2. The Weak Lefschetz Property for Artinian Modules

Throughout this section, let $\mathbb{K}$ be an algebraically closed field, $S = \mathbb{K}[x_0, \ldots, x_r]$, and $\mathfrak{m}$ the ideal $(x_0, \ldots, x_r)$. Any S-module considered will be finitely generated. In particular, any Artinian S-module has finite length.

An Artinian S-module $N = \bigoplus_{j \in \mathbb{Z}} N_j$ has the *Weak Lefschetz Property* (WLP) if given a *general* linear form ℓ, the $\mathbb{K}$-linear map $\times \ell : N_j \longrightarrow N_{j+1}$ has maximal rank for all j. The following question is the leading motivation for this project:

Question 3. Which Artinian S-modules have the Weak Lefschetz Property?

This a broad question, and is unresolved even in codimension three for Artinian Gorenstein algebras. The most complete result to date is (Theorem 2.3, [19]), which says that if $\mathbb{K}$ has characteristic zero, $r = 2$, and $N = S/I$, with I a complete intersection, then N has the WLP. The perspective in [14] was that N need not be cyclic, and the main result of [19] was not only generalized, but the proof is self-contained. Noting that in [19], the proof of the main theorem requires results from [36]. Our result is the following:

Theorem 2 (Theorem 3.7, [14])**.** Suppose $\mathbb{K}$ has characteristic zero. If $\mathcal{E}$ is a rank two vector bundle on $\mathbb{P}^2$, then $H^1_*\left(\mathbb{P}^2, \mathcal{E}\right)$ has the WLP.

For ease, set $M := H^1_*\left(\mathbb{P}^2, \mathcal{E}\right)$, with $\mathcal{E}$ as in Theorem 2. As observed in [14], the minimal free resolution of M played an important role in determining if M had the WLP. In particular, the minimal free resolution of M is given by the Buchsbaum-Rim complex, and this minimal free resolution is symmetric, as are minimal free resolutions of Gorenstein algebras. In [16], this is explored further by studying *Symmetrically Gorenstein* modules. Symmetrically Gorenstein modules were first defined and studied in [24], and are generalizations of Artinian Gorenstein algebras.

Definition 1 (Definition 3.2, [24])**.** A graded Artinian S-module M is *Symmetrically Gorenstein* if there is an isomorphism $\tau : N \longrightarrow \mathrm{Hom}_{\mathbb{K}}(N, \mathbb{K})(-s)$ such that $\tau = \pm\mathrm{Hom}_{\mathbb{K}}(\tau, \mathbb{K})(-s)$.

The module $H^1_*\left(\mathbb{P}^2, \mathcal{E}\right)$ from Theorem 2 is an example of a Symmetrically Gorenstein module in codimension three, as noted in (Proposition 3.9, [16]). The initial interest in showing M was Symmetrically Gorenstein was to generalize (Theorem 2.3, [19]), however, this property turned out to be very useful in allowing me to generalize geometric results from [4] on the *non-Lefschetz locus* of an Artinian S-module, as well as providing interesting connections with Artinian *level* modules from [3]. For example, in [16], the following is shown:

Theorem 3 (Corollary 5.8, [16])**.** Suppose $\mathbb{K}$ has characteristic zero and N is a nonnegatively graded level S-module. Then the non-Lefschetz locus of N can be defined by two degrees.

Here, a *level* S-module is a nonnegatively graded Artinian S-module which is assumed to be shifted to be of the form $N = N_0 \oplus \cdots \oplus N_c$, for which $\mathrm{Soc}(N) = (0 :_N \mathfrak{m}) = N_c$. The non-Lefschetz locus of N, denoted by $\mathcal{L}_N$, is a scheme attached to N, which, as a set, is a union of $\mathcal{L}_{N,j}$, for $0 \leq j \leq c$, with $\mathcal{L}_{N,j} := \{[\ell] \in \mathbb{P}(S_1) | \times \ell : N_j \longrightarrow N_{j+1} \text{ does not have maximal rank}\}$ (where, if $\ell = a_0x_0 + \cdots a_rx_r$, $[\ell] = [a_0 : \cdots : a_r] \in \mathbb{P}^r$). The proof of Theorem 3 also required the generalization of the crucial result (Proposition 2.1, [28]), as well as generalizations of important results from [3].

Moreover, utilizing the groundwork that has been laid, it is shown that the non-Lefschetz locus of a nonnegatively-graded Symmetrically Gorenstein S-module can be defined by a single degree (Proposition 5.11, [16]), generalizing (Corollary 2.7, [4]).

Goal 2. Study the WLP for classes of Symmetrically Gorenstein S-modules utilizing the framework and techniques that were laid out in [16]. Generalize results from [14] to higher codimension.

The plan to accomplish Goal 2 is to provide an analysis of syzygy bundles (which are sheafifications of syzygies of a module) of Symmetrically Gorenstein S-modules with the goal of using this analysis to characterize when certain classes of Symmetrically Gorenstein S-modules have the WLP. Significant information about the WLP can be gained from studying syzygy bundles, as can be seen in [1, 6, 8, 10, 11, 19]. Moreover, in [14], the importance of the syzygy bundle is again recognized for non-cyclic S-modules in codimension three, highlighting the use of the Grauert-Mülich theorem on the generic splitting type of a semistable bundle of rank two on $\mathbb{P}^2$ (Corollary 2, pg. 206, [33]). Moreover, we fully determine the generic splitting type for unstable syzygy bundles of rank two on $\mathbb{P}^2$ in [14] (see Proposition 3.5, [14]), which is something that had not been given attention to previously in this context.

I am also interested in expanding applications of the the aforementioned theorem of Grauert-Mülich, noting it applies more generally to semistable rank r bundles on $\mathbb{P}^r$ (Corollary 1, pg. 205, [33]). In particular, there are only finitely many generic splitting types for a semistable bundle of rank r on $\mathbb{P}^r$. To make use of this, the semistability of a bundle may need to be determined, but in this situation, a theorem of Bonhorst and Spindler (Theorem 2.7, [5]) can be used, as well as computational methods from [23].

Moreover, this analysis of the syzygy bundle will provide insight into generalizing results on the non-Lefschetz locus from [4] (see, for example, the proof of Theorem 5.3 in [4]). In particular, insight into geometric objects such as the non-Lefschetz locus for non-cyclic Artinian modules will allow for comparison to the structure of the non-Lefschetz locus for Artinian algebras, and insight into when Artinian algebras have the WLP.

3. Duals of Hyperplane Arrangements

Let $\mathbb{K}$ be an algebraically closed field of characteristic zero and $S = \mathbb{K}[x_0, \ldots, x_r]$. A *hyperplane arrangement* is a finite union of hyperplanes in $\mathbb{P}^r$. Given a hyperplane arrangement $\mathcal{A}$, denote its defining polynomial by $f_\mathcal{A} \in S$, noting that $f_\mathcal{A}$ is a product of the linear forms that define $\mathcal{A}$. Hyperplane arrangements are well-studied objects, and have deep connections to algebra, combinatorics, and topology (see [30, 35]). The following class of hyperplane arrangements is of interest to us:

Definition 2. A hyperplane arrangement $\mathcal{A}$ in $\mathbb{P}^r$ is called *generic* if given a subcollection $\mathcal{A}' \subseteq \mathcal{A}$ with $|\mathcal{A}'| = p$,

(a) the linear forms that define the hyperplanes in $\mathcal{A}'$ are linearly independent if $p \leq r + 1$;
(b) the intersection of the hyperplanes in $\mathcal{A}'$ is empty if $p > r + 1$.

Set $R = \mathbb{K}[X_0, \ldots, X_r]$ and let R act on S by partial differentiation. Given $f \in S$, write $f^\perp$ for the ideal of R given by $\text{Ann}_R(f)$. For any $f \in S$, it is well-known that the quotient $f^\perp$ is an Artinian Gorenstein ideal (that is, the quotient $R/f^\perp$ is an Artinian Gorenstein algebra). A natural question to ask is when do the ideals $f^\perp$ define complete intersections? This leads to the main question:

Question 4. Given a generic hyperplane arrangement $\mathcal{A}$, determine conditions on $\mathcal{A}$ so that $f_\mathcal{A}^\perp$ is a complete intersection?

Techniques for studying the structure of $f^\perp$ are usually developed for specific classes of forms, so are not broadly applicable. In fact, in (Ch. 9, Section L., [21]), it is noted that the problem of determining conditions on f so that $f^\perp$ is a complete intersection is quite open. One example of answering this question comes, albeit indirectly, from [7], where annihilators of forms called *direct sums* are studied. We note direct sums vary significantly from products of linear forms (Proposition 2.12, [7]).

A natural place to begin to answer Question 4 is to give a lower bound on $\alpha(f_\mathcal{A}^\perp)$, the minimal degree of a polynomial in $f_\mathcal{A}^\perp$. Utilizing results on star configurations from [17], the following is shown:

Theorem 4. (Proposition 4.10, [12]) If $\mathcal{A}$ is a generic hyperplane arrangement in $\mathbb{P}^r$ with at least $r + 1$ hyperplanes, then $\alpha(f_\mathcal{A}^\perp) \geq \min\{|\mathcal{A}| - r + 1, r + 1\}$

Using the above, we obtain a partial answer to Question 4:

Corollary 1. (Corollary 4.11, [12]) If $\mathcal{A}$ is a generic hyperplane arrangement in $\mathbb{P}^r$ with at least $r + 2$ hyperplanes, and $f_\mathcal{A}^\perp$ is a complete intersection, then $|\mathcal{A}| \geq r(r + 1)$

Note when $r = 2$, Corollary 1 fails to give any information about whether or not $f_{\mathcal{A}}^{\perp}$ is a complete intersection. We investigate this specific case below.

Example 1. (Theorem 5.1(*), [12])
Let $\mathbb{K} = \mathbb{C}$, $r = 2$, and $\omega = \exp(\frac{\pi i}{3})$. If $f_{\mathcal{A}} = xyz(x+y+z)(x+\omega y+\overline{\omega})(x+\overline{\omega}y+\omega z)$. Then $f_{\mathcal{A}}^{\perp}$ contains $X^3-Y^3, X^3-Z^3, XY^2+YZ^2+ZX^2, X^2Y+Y^2Z+XZ^2$. As $f_{\mathcal{A}}^{\perp}$ contains no quadrics, these are part of a minimal generating set for $f_{\mathcal{A}}^{\perp}$, so that $f_{\mathcal{A}}^{\perp}$ is not a complete intersection.

As the above example is not a complete intersection, this leads to further investigate this situation and to the next goal of this project:

Goal 3. Show that when $r = 2$, $\mathbb{K} = \mathbb{C}$, and $|\mathcal{A}| = 6$, $f_{\mathcal{A}}^{\perp}$, is not a complete intersection by classifying all such generic hyperplane arrangements in $\mathbb{P}^2$. So far, there is no rigorous proof of this, but there is a numerical and computational classification from the use of MACAULAY2 [18] and BERTINI [2] (see Theorem 5.1(*) in [12]).

There is also interesting connection with *Waring Rank* from [7]: The *Waring rank* of $f \in S_d$ is the smallest s for which $f = \ell_1^d + \cdots + \ell_s^d$, with the $\ell_i \in S_1$ pairwise linearly independent. With $\mathcal{A}$ as in Example 1, the ideal generated by the four cubics in Example 1 defines six distinct points in $\mathbb{P}^2$, so that by the well-known Apolarity Lemma (see [21] for a proof), $f_{\mathcal{A}}$ has Waring rank six. In fact, $f_{\mathcal{A}}$ is a generic example with minimal Waring Rank, which is often unexpected.

In this direction, we also have the following corollary from Theorem 4:

Corollary 2. If $\mathcal{A}$ is a generic hyperplane arrangement in $\mathbb{P}^r$ with at least $r+1$ hyperplanes, the Waring rank of $f_{\mathcal{A}}$ is at least $\min\left\{\binom{|\mathcal{A}|}{r}, \binom{2r}{r}]\right\}$.

References

[1] Abe T. and Dimca A., Splitting types of bundles of logarithmic vector fields along plane curves, *International Journal of Mathematics*, 29 (2018)

[2] Bates D., Hauenstein J., Sommese A., Wampler C.,Bertini: Software for Numerical Algebraic Geometry, Available at `bertini.nd.edu` with permanent doi: dx.doi.org/10.7274/R0H41PB5

[3] Boij M., Artin Level Modules, *Journal of Algebra*, 226 (2000), 361-374

[4] Boij M., Migliore J., Miró-Roig R.M., and Nagel U., The non-Lefschetz Locus, *Journal of Algebra*, 505 (2018), 288-320

[5] Bohnhorst G. and Spindler H., , The Stability of Certain Vector bundles on $\mathbb{P}^n$, *Complex Algebraic Varieties*, (1992), 39-50

[6] Brenner H. and Kaid A., Syzygy Bundles on $\mathbb{P}^2$ and the Weak Lefschetz Property, *Illinois Journal of Mathematics*, 4 (2007), 1299-1308

[7] Buczyńska W., Buczyński J., Kleppe J., Teitler Z., Apolarity and Direct Sum Decomposability of Polynomials, *Michigan Mathematical Journal*, 64 (2015), 675-719

[8] Cook D. and Nagel U., Syzygy Bundles and the Weak Lefschetz Property of Almost Complete Intersections, *Journal of Commutative Algebra*, (2019)

[9] Dao H. and Huneke C., Vanishing of Ext, Cluster Tilting modules and Finite Global Dimension of Endomorphism Rings, *American Journal of Mathematics*, 135 (2013), 561-578

[10] Dimca A. and Popescu D., Hilbert Series and Lefschetz Properties of Dimension One Almost Complete Intersections, *Communications in Algebra*, 44 (2016), 4467-4482

[11] Dimca A. and Sticlaru G. On the Jumping Lines of Bundles of Logarithmic Vector Fields Along Plane Curves, `https://arxiv.org/abs/1804.06349`

[12] DiPasquale, M., Flores, Z., Peterson, C., The Apolar Algebra of a Product of Linear Forms (Accepted to the *International Symposium on Symbolic and Algebraic Computation*)

[13] Dugger D. and Shipley B., K-theory and Derived Equivalences, *Duke Mathematics Journal*, 124 (2004), 587-617

[14] Failla G., Flores Z., Peterson C., On the Weak Lefschetz Property for Vector Bundles on $\mathbb{P}^2$, `https://arxiv.org/abs/1803.10337`

[15] Flores Z., G-Groups of Cohen-Macaulay Rings with n-Cluster Tilting Objects, *Algebras and Representation Theory* (2019), `https://doi.org/10.1007/s10468-019-09876-6`

[16] Flores Z., Symmetry, Unimodality, and Lefschetz Properties for Graded Modules, https://arxiv.org/abs/1908.03648

[17] Geramita A.V., Harbourne B., Migliore J., Star Configurations in $\mathbb{P}^n$, *Journal of Algebra* 376 (2013), 279299

[18] Grayson D. and Stillman M., Macaulay2, a software system for research in algebraic geometry, available at ttp://www.math.uiuc.edu/Macaulay2/

[19] Harima T., Migliore J.C., Nagel U., Watanabe J., The Weak and Strong Lefschetz Properties for Artinian K-algebras, *Journal of Algebra*, 262 (2003), 99-126

[20] Holm H., K-Groups for Rings of Finite Cohen-Macaulay Type, *Forum Mathematicum*, 27 (2015), 2413-2452

[21] Iarrobino A. and Kanev V., *Power Sums, Gorenstein Algebras, and Determinantal Loci*, Lecture Notes in Mathematics, Springer-Verlag Berlin Heidelberg, 1999

[22] Iyama O., *Trends in Representation Theory of Algebras and Related Topics*, Auslander-Reiten Theory Revisited, European Mathematical Society, Zürich,(2008), 349-398.

[23] Kaid A. and Kasprowitz R., Semistable Vector Bundles and Tannaka Duality from a Computational Point of View, *Experimental Mathematics*, 21 (2012), 171-188

[24] Kunte M., Gorenstein Modules of Finite length, *Mathematische Nachrichten*, 284 (2011), 899-919

[25] Lee R. and Szczarba R.H., The group $K_3(\mathbb{Z})$ is Cyclic of Order 48, Annals of Mathematics, 104 (1976), 3160

[26] Leuschke G.J., Endomorphism Rings of Finite Global Dimension, *Canadian Journal of Mathematics*, 59 (2007), 332-342

[27] Leuschke G.J. and Wiegand R., *Cohen-Macaulay Representations*, American Mathematical Society, Providence, RI (2012).

[28] Migliore J.C., Miró-Roig R.M., and Nagel U., Monomial Ideals, Almost Complete Intersections, and the Weak Lefschetz Property, *Trans. Amer. Math. Soc.*, 363 (2011), 229-257

[29] Nebjosa P. and Shinder E., K-theory and the Singularity Category of Quotient Singularities https://arxiv.org/abs/1809.10919

[30] Orlik P. and Solomon L., Combinatorics and Topology of Complements of Hyperplanes, *Inventiones Mathematica*, 56 (1980), 167-189

[31] Navkal V. (2013), *K'-theory of a Cohen-Macaulay Local Ring with n-Cluster Tilting Object* (Doctoral dissertation), Retrieved from ProQuest Dissertations and Theses (Order No. 3563356)

[32] Navkal V., K'-Theory of a Local Ring of Finite Cohen-Macaulay Type, *Journal of K-theory*, 12 (2013), 405-432.

[33] Okonek C., Schneider M., and Spindler H., *Vector Bundles on Complex Projective Space*, Birkhäuser Boston, Boston, MA, 1980.

[34] Quillen D., *Algebraic K-Theory I. Proceedings of the Conference Held at the Seattle Research Center of Battelle Memorial Institute, August 28 - September 8, 1972*, Higher Algebraic K-theory I, 85-147, Springer-Verlag, Berlin (1973).

[35] Terao H., Generalized Exponents of a Free Arrangement of Hyperplanes and Shepherd-Todd-Brieskorn Formula, *Inventiones mathematicae*, 63 (1981), 159-179

[36] Watanabe J., A Note on Complete Intersections of Height Three, *Proceedings of the American Mathematical Society* 126 (1998), 31613168.